# Dead Ends in the Heisenberg Group

Noah MacAulay

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In this note, we prove that the Cayley graph of the Heisenberg group with standard generators contains an infinite number of dead ends. First, some definitions.

A dead end is an element $d$ of a finitely presented group $G$ such that, for all neighbours $v$ of $d$, $l(v) \leq l(d)$ where $l$ is the word norm on $G$.

The Heisenberg group has the following presentation:

$$< x, y, t | xy = yx, xt = tx, yty^{-1}t^{-1} = x >$$

To prove that the Heisenberg group contains an infinite number of dead ends, we will consider elements of the form $x^n$. First we will prove that, for all sufficiently large $c$, $x^c y$, $x^c t$, $x^c y^{-1}$, and $x^c t^{-1}$, have length less than or equal to $x^c$.

There exists a number $c_0$ such that, for all $c \geq c_0$, the shortest word for $x^c$ must contain a certain number of $y$s and $t$s. As $x$ commutes with $y$ and $t$, we can write this word as

$$x^n Q$$

where Q is an expression containing only $y$s and $t$s. Since $Q$ can ultimately be reduced to an expression of the form $x^n$, this implies that $Q = yQy^{-1}$, and $Q = tQt^{-1}$. Since $Q$ must contain some elements of the form $t$, $t^{-1}$, $y$, $y^{-1}$, we can use these relations to write

$$Q = Q_0 y = Q_1 y^{-1} = Q_2 t = Q_3 t^{-1}$$

without changing the length of $Q$. This implies that $x^c y$, $x^c t$, $x^c y^{-1}$, and $x^c t^{-1}$, have length less than $x^c$.

Also note that $x^c x^{-1}$ evidently has length less than or equal to that of $x^c$.

Now consider words of the form $x^{n^2}$ for some $n$. These elements have geodesic words of the form $y^n t^n y^{-n} t^{-n}$, of length $4n$. Consider two such elements, $x^{j^2}$ and $x^{(j+1)^2}$. The number of elements between these two words is $2j$, but the difference of their word length is only 4. This implies that between them, there must be a dead end. Hence, the Heisenberg group contains an infinite number of dead ends.